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Ihor LYASHENKO,
Yuriy TADEYEV

OPTIMUM TRAJECTORY OF INTER-BRANCH MODEL OF ECOLOGIC-ECONOMICAL DEVELOPMENT

Abstract

This paper analyses a question of expansion of classic dynamic inter-branch model in case of the ecological-economic system. There is main trajectory of the balanced development of Neyman built for the offered dynamic inter-branch ecological-economic model.

Key words:

Ecological-economic system, inter-branch ecological-economic model, dynamic equilibrium, main trajectory, maximal rate of the economic growth, the Neyman ray.

Lyashenko Ihor, Doctor of Physics and Mathematical Sciences, Professor, Kyiv National University by T. Shevchenko, Ukraine.
Tadeyev Yuriy, Kyiv National University by T. Shevchenko, Ukraine.

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For the first time an inter-branch ecological-economic model was introduced by V. Leontief and D. Ford [1]:

$$\begin{aligned} x_1 &= A_{11}x_1 + A_{12}x_2 + y_1 \\ x_2 &= A_{21}x_1 + A_{22}x_2 - y_2 \end{aligned}, \qquad (1)$$

Where, x_1 – vector of product issue, x_2 – vector of pollutants disposal, y_1 – vector of finished products, y_2 – vector of discharge of pollutants into environment, A_{11} and A_{12} – matrices of material expenses, A_{21} and A_{22} – matrices of pollutants issue.

A relevant diversified model of ecological-economic development is related to the largeness of optimization task, that is an obstacle to the practical use. Similar tasks can be successfully solved through high-quality methods of the optimum trajectory research.

It is shown in the theory of the growing economy by Dorfman, Samuelson and Solow [2], that the effective trajectory of the economic growth has a long-term tendency to approach the Neyman's way of the proof balanced growth (highways).

We are going to show basic concepts and conclusions of highway theory on the example of optimization task for the Neyman's model [3,4]:

$$\begin{aligned} & C_T x_T \to \max \\ & Ax_t \le Bx_{t-1}, \quad t = 1,2,\ldots,T, \end{aligned} \qquad (2)$$

Where A and B – inalienable rectangular matrices of charges and issue, C_T – positively set vector, x_T – vector of intensity of technological processes in the time t.

The stationary balanced trajectory for a model (2) is determined by the rate of growth $\bar{\lambda}^{-1}$ and the Neyman's ray $\bar{x}$ and looks like $x_t = \bar{\lambda}^{-t}\bar{x}$, where $\bar{\lambda} > 0$, $\bar{x} > 0$ is the only unique decision of the system of inequality

$$A\bar{x} \le \bar{\lambda} B\bar{x}. \qquad (3)$$

It appears that highway $\bar{x}$ is less sensitive to the change of coefficients of having a special purpose functional C_T and a task (2) is taken to such task of Neyman

$$\lambda \to \min, \quad Ax \le \lambda Bx, \quad x \ge 0. \qquad (4)$$

The basic result with regard to the Neyman's model (4) consists of such an assertion [3, 4]:

The assertion 1. *Let inalienable matrices A, B – such, that matrix of issue B does not have zeroing rows, and matrix of charges A does not have zeroing columns. So, then the irresolvable productive Neyman's model (4) has the unique rate of growth $\overline{\lambda} < 1$ and highway $\overline{x} > 0$.*

Here the model irresolvableness means that the number of Fronbenius model is simple, and the Fronbenius vector strictly positive. Model productivity means that the Fronbenuis number is less than 1.

One of the most known schemes of dynamic inter-branch balance is π-model developed by Y. P. Ivanilov and O. O. Petrov [5]:

$$\begin{gathered} Ax_t + D\eta_t + L_t c \le x_t, \\ x_t \le \xi_{t-1}, \quad \xi_t \le \xi_{t-1} + \eta_t, \\ lx_t \le L_t, \\ (x_t, \xi_t, \eta_t, L_t) \ge 0, \quad t = 1,2,\ldots,T, \end{gathered} \tag{5}$$

Where ξ_t – power vector of the products issue, η_t – growth vector of powers, D – inevitable matrix of material expenses, l –labour intensive vector of the products issue, L_t – total number of workers, c – vector of per capita consumption (actual payment of worker).

A different criterion of the optimum functioning of economy are added to the model (5), in particular terminal criterion

$$c_T x_T \to \max, \tag{6}$$

Where $c_T > 0$ has a link with the maximum pace of economic growth.

The basic results in relation to existence of the highway for a task (5)–(6) is formulated as such assertion [3]:

The assertion 2. *If $\xi_0 > 0$, matrix $R = (c_i l_j)_1^n$, matrix A+R is irresolvable and productive, matrix $Q(\lambda) = \lambda(A+R) + (1-\lambda)D$ is primitive. So, then vector l s$(\overline{x}, \overline{\xi}, \overline{\eta}, \overline{L})$, where $\lambda = \overline{\lambda} < 1$ and $\overline{x} > 0$ accordingly, the Frobenius number and the Frobenuis vector of $Q(\lambda)$ matrix, is a highway for the model* (5)–(6).

Today, there is an unsolved question of expansion π-model (4) in case of the ecological-economic system and establishment of highway development of this an ecological-economic system. The paper is about this problem.

The following model is given:

$$
\begin{gathered}
c_1^T x_1(T) + c_2^T x_2(T) \to \max, \\
A_{11}x_1(t) + A_{12}x_2(t) + D_{11}\eta_1(t) + D_{12}\eta_2(t) + c_1 L(t) \le x_1(t), \\
A_{21}x_1(t) + A_{22}x_2(t) + D_{21}\eta_1(t) + D_{22}\eta_2(t) + c_2 L(t) \le x_2(t) + y_2(t), \\
x_1(t) \le \xi_1(t-1), \quad x_2(t) \le \xi_2(t-1), \\
\xi_1(t) \le \xi_1(t-1) + \eta_1(t), \quad \xi_2(t) \le \xi_2(t-1) + \eta_2(t), \\
l_1 x_1(t) + l_2 x_2(t) \le L(t), \\
y_2(t) \le H_1 x_1(t) + H_2 x_2(t), \\
x_1(t) \ge 0, \quad x_2(t) \ge 0, \quad \xi_1(t) \ge 0, \quad \xi_2(t) \ge 0, \quad \eta_1(t) \ge 0, \quad \eta_2(t) \ge 0, \\
L(t) \ge 0, \quad y_2(t) \ge 0, \quad t = 1,2,\dots,T,
\end{gathered} \tag{7}
$$

$\xi_1(0) > 0, \quad \xi_2(0) > 0$ – are given.

Here, in addition to the model (1) are, $\xi_1(t)$ – power vector for the products issue, $\xi_2(t)$ – power vector for pollutants dispose, $\eta_1(t)$ – power vector increase of producing industries, $\eta_2(t)$ – vector of power increase of purifying devices, D_{11}, D_{12} – inalienable matrices of material charges, D_{21}, D_{22} – inalienable matrices of pollutants issue at construction of basic production and purifying devices, $c_1 > 0$ – vector of actual payment of one worker-producer, $c_2 > 0$ – vector of issue of domestic contaminations per one worker-producer, $l_1 > 0$ – vector of labour intensive of products issue, $l_2 > 0$ – vector of labour intensive of pollutants dispose, *L(t)* – total number of workers, and H_1, H_2 – inalienable matrices of the technological outliers of pollutants into environment by basic production and purifying devices.

Let us explore a state of equilibrium for the system (7). The relevant stationary trajectory of intensity is determined by the rate of growth $\lambda^{-1} > 1$ by the Neyman's ray $X = (x_1, x_2, \xi_1, \xi_2, \eta_1, \eta_2, y_2, L)$ and has a following look

$$
\begin{gathered}
x_1(t) = \lambda^{-t}x_1, \quad x_2(t) = \lambda^{-t}x_2, \quad \xi_1(t) = \lambda^{-t}\xi_1, \quad \xi_2(t) = \lambda^{-t}\xi_2, \\
\eta_1(t) = \lambda^{-t}\eta_1, \quad \eta_2(t) = \lambda^{-t}\eta_2, \quad y_2(t) = \lambda^{-t}y_2, \quad L(t) = \lambda^{-t}L.
\end{gathered} \tag{8}
$$

If we correlate (8) with (7), so for the state of equilibrium $(\lambda, x_1, x_2, \xi_1, \xi_2, \eta_1, \eta_2, y_2, L)$ at large *T* we get optimization task

$$\lambda \to \min,$$
$$x_1 \geq A_{11}x_1 + A_{12}x_2 + D_{11}\eta_1 + D_{12}\eta_2 + Lc_1,$$
$$x_2 \geq A_{21}x_1 + A_{22}x_2 + D_{21}\eta_1 + D_{22}\eta_2 + Lc_2 - y_2,$$
$$x_1 \leq \lambda\xi_1, \quad x_2 \leq \lambda\xi_2, \quad (1-\lambda)\xi_1 \leq \eta_1, \quad (1-\lambda)\xi_2 \leq \eta_2, \tag{9}$$
$$l_1x_1 + l_2x_2 \leq L,$$
$$y_2 \leq H_1x_1 + H_2x_2,$$
$$x_1 \geq 0, \quad x_2 \geq 0, \quad \xi_1 \geq 0, \quad \xi_2 \geq 0, \quad \eta_1 \geq 0, \quad \eta_2 \geq 0, \quad L \geq 0, \quad y_2 \geq 0.$$

We are going to examine matrices

$$R_{11} = \left(c_i^1 l_j^1\right), \quad R_{12} = \left(c_i^1 l_j^2\right), \quad R_{21} = \left(c_i^2 l_j^1\right), \quad R_{22} = \left(c_i^2 l_j^2\right).$$

As at $0 < \lambda < 1$ we have

$$x_1 \leq \lambda\xi_1 \leq \frac{\lambda}{1-\lambda}\eta_1, \quad x_2 \leq \lambda\xi_2 \leq \frac{\lambda}{1-\lambda}\eta_2,$$

So

$$\eta_1 \geq \frac{\lambda}{1-\lambda}x_1, \quad \eta_2 \geq \frac{\lambda}{1-\lambda}x_2,$$

Taking into consideration that $D_{11} \geq 0, \quad D_{12} \geq 0, \quad D_{21} \geq 0, \quad D_{22} \geq 0,$ we get

$$D_{11}\eta_1 \geq \frac{1-\lambda}{\lambda}D_{11}x_1, \quad D_{12}\eta_2 \geq \frac{1-\lambda}{\lambda}D_{12}x_2,$$
$$D_{21}\eta_1 \geq \frac{1-\lambda}{\lambda}D_{21}x_1, \quad D_{22}\eta_2 \geq \frac{1-\lambda}{\lambda}D_{22}x_2.$$

Because

$$(l_1x_1)c_1 = R_{11}x_1, \quad (l_2x_2)c_1 = R_{12}x_2,$$
$$(l_1x_1)c_2 = R_{21}x_1, \quad (l_2x_2)c_2 = R_{22}x_2,$$

So taking into account inalienability of matrices H_1 and H_2, we find such inequalities,

$$x_1 \geq A_{11}x_1 + A_{12}x_2 + D_{11}\eta_1 + D_{12}\eta_2 + Lc_1 \geq \left(A_{11} + \frac{1-\lambda}{\lambda}D_{11}\right)x_1 + \left(A_{12} + \frac{1-\lambda}{\lambda}D_{12}\right)x_2 +$$
$$+ (l_1x_1 + l_2x_2)c_1 \geq \left(A_{11} + R_{11} + \frac{1-\lambda}{\lambda}D_{11}\right)x_1 + \left(A_{12} + R_{12} + \frac{1-\lambda}{\lambda}D_{12}\right)x_2,$$
$$x_2 \geq A_{21}x_1 + A_{22}x_2 + D_{21}\eta_1 + D_{22}\eta_2 + Lc_2 - y_2 \geq$$
$$\geq \left(A_{21} + R_{21} + \frac{1-\lambda}{\lambda}D_{21}\right)x_1 + \left(A_{22} + R_{22} + \frac{1-\lambda}{\lambda}D_{22}\right)x_2 - H_1x_1 - H_2x_2.$$

The last can be written in the following way

$$x_2 + H_1 x_1 + H_2 x_2 \geq \left(A_{21} + R_{21} + \frac{1-\lambda}{\lambda} D_{21} \right) x_1 + \left(A_{22} + R_{22} + \frac{1-\lambda}{\lambda} D_{22} \right) x_2.$$

After multiplication of both parts of the obtained inequalities to $\lambda > 0,$ we get

$$\lambda(E + H)X \geq [\lambda(A + R) + (1-\lambda)D]X, \qquad (10)$$

Where $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$ $R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix},$ $D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix},$ $H = \begin{pmatrix} 0 & 0 \\ H_1 & H_2 \end{pmatrix}$ – inalienable square matrices, E is a single matrix, $X = (X_1, X_2)^T$ – is a vector-column.

Thus, the task of growth rate maximization is generalized to the Neyman's model

$$\lambda \to \min, \quad Q(\lambda)x \leq \lambda Bx, \quad x \geq 0, \qquad (11)$$

Where $$Q(\lambda) = \lambda(A + R) + (1-\lambda)D, \quad B = E + H. \qquad (12)$$

For the (11) we build an ambivalent task

$$pQ(\lambda) \geq \lambda pB, \quad p \geq 0, \qquad (13)$$

Where $p = (p_1, p_2)$ – row vector of ambivalent estimations. Due to our interest in $x > 0,$ so then it is possible only, when

$$pQ(\lambda) = \lambda pB. \qquad (14)$$

The system of linear algebraic equations (14) has a decision $p \neq 0$ only if

$$\det(Q(\lambda) - \lambda B) = 0, \qquad (15)$$

That is

$$\det[(1-\lambda)D - \lambda(E - A - R + H)] = 0. \qquad (16)$$

Let matrix $A + R$ is productive, then [3, 4] exists $(E - A - R)^{-1} \geq 0.$

Let matrix H also to be small that $H \leq A + R$ (as such it is carried out, because $A \geq 0,$ $R > 0$). Meanwhile, matrix $\overline{A} = A + R - H \geq 0$ remains productive, so exist $(E - \overline{A})^{-1} = (E - A - R + H)^{-1} \geq 0.$

The equation (16) might be written as following:

$$\det\left[(E - A - R + H)^{-1} D - \mu E\right] = 0, \qquad (17)$$

Where $\mu = \frac{\lambda}{1-\lambda} = \frac{1}{1-\lambda} - 1 > 0$ under $0 < \lambda < 1$. to the least value of λ belongs the highest value μ.

The matrix is $(E - A - R + H)^{-1} \geq 0$. Then, according to the Perron-Frobenius theorem [4] there is Frobenius number $\bar{\mu} > 0$ and corresponding Frobenius vector $\bar{z} \geq 0$, so that

$$(E - A - R + H)^{-1} D\bar{z} = \bar{\mu}\bar{z}. \qquad (18)$$

We underline here, that

$$(E - A - R + H)^{-1} D \leq (E - A - R)^{-1} D,$$

and that is why $\bar{\mu} \leq \mu^*$, where μ^* – is the Frobenius number of $(E - A - R)^{-1} D$ matrix.

Now let us get back to the task (11), which can be written as following

$$\mu \to \min, \quad (E - A - R + H)^{-1} x \leq \mu x, \quad x \geq 0. \qquad (19)$$

A solution of this task is simply achieved by $\mu = \bar{\mu}, \quad x = \bar{z}$. Thereat the pace of growth is $\bar{\lambda}^{-1} = 1 + \frac{1}{\bar{\mu}} \geq 1 + \frac{1}{\mu^*} > 1$, and also the structure of production is $\bar{x} \geq 0$.

This is a time now to find out about a state of equilibrium of ecological-economic model (7). The equilibrium trajectory of intensity is determined by the rate of growth λ^{-1} and the Neyman's ray $X = (x_1, x_2, \xi_1, \xi_2, \eta_1, \eta_2, y_2, L)$. In order to provide existence of unbanal decision of the inequalities system (9), it is necessary to impose some terms on the model parameters.

Assume that matrix A is an inalienable and irresolvable of $l > 0, \quad c > 0$, and matrix $A + R$ is productive, hence its Frobenius number is less than 1. In this content, this limitation consists in the existent technology (A, l) and allows for every worker «to feed» himself, carrying out a production process (producing products and get rid of contaminations).

Besides, we assume that $D \geq 0$ and if $\eta \geq 0, \quad D\eta = 0$, so than $\eta = 0$. Such a supposition means that any increase of production power and purifying devices requires a certain material expenses. In other words, we consider that there are no zeroing columns in the D matrix.

The main conclusion of this paper is formulated as following assertion.

The assertion 3. *If matrix $A+R$ is productive, so it is $H \le A+R$, and matrix D has no zeroing columns, thus the model* (7) *has an equilibrium state with the pace of growth $\bar{\lambda}^{-1} = 1+\frac{1}{\bar{\mu}}$, where there is only corresponding Neyman's ray $(x_1, x_2, \xi_1, \xi_2, \eta_1, \eta_2, y_2, L)$, at*

1) $\bar{\mu}$ – *the Frobenius number,* $\bar{x}$ – *the Frobenius vector of* $(E-A-R+H)^{-1}D$ *matrix;*

2) $$\bar{\xi} = \bar{\lambda}^{-1}\bar{x}, \quad \bar{\eta} = \frac{1-\bar{\lambda}}{\bar{\lambda}}\bar{x}, \quad \bar{y}_2 = H_1\bar{x}_1 + H_2\bar{x}_2, \quad \bar{L} = l\bar{x}. \qquad (20)$$

The evidence. Let us take a problem

$$\begin{gathered} \lambda \to \min, \\ (\lambda(A+R-H)+(1-\lambda)D)x \le \lambda x, \quad ex = 1, \quad x \ge 0, \end{gathered} \qquad (21)$$

Where $e = (1,1,\ldots,1)$.

According to evidence the Frobenius number for a matrix $A+R$ is less than 1. As $H \le A+R$, the Frobenius number for a matrix $A+R-H$ is also less than 1. At the same time, from terms on a matrix *D* follows, that the Frobenius number for a matrix *D* is more than 0. The task (21) has a solution $(\bar{x},\bar{\lambda})$, where $0<\bar{\lambda}<1$. We indicate that number $\bar{\lambda}^{-1}$ is the pace of growth for the model (7), to which corresponds the Neyman's ray. It is obvious that vector $(\bar{x},\bar{\xi},\bar{\eta},\bar{y}_2,\bar{L})$, which is determined by equalizations (20), is the solution to the system of inequalities (9).

Now (x,ξ,η,y_2,L) – be an arbitrary non-zero vector that satisfies the system of inequalities (9), and is a decision of this system at $\lambda = \bar{\lambda}$. It can be shown that from the terms $0<\bar{\lambda}<1$ and $(x,\xi,\eta,y_2,L) \ne 0$ follows, that $x \ne 0$. From the system (9) we have such inequalities

$$\frac{1-\bar{\lambda}}{\bar{\lambda}}x \le (1-\bar{\lambda})\xi \le \eta, \qquad (22)$$

from where with $D \ge 0$, we get

$$\frac{1-\bar{\lambda}}{\bar{\lambda}}D\,x \le D\,\eta. \qquad (23)$$

Just because $Rx = lxc$, from (9) and (23) we have

$$\left(A + R - H + \frac{1-\bar{\lambda}}{\bar{\lambda}} D\right) x \le x \qquad (24)$$

or

$$(E - A - R + H)^{-1} D x \ge \bar{\mu} x. \qquad (25)$$

As $\bar{\mu}$ – is the Frobenius number for $(E - A - R + H)^{-1} D,$ matrix, so it is clear that (25) can be true only when x – the Frobenius vector. It means that in (25) and in (23) inequalities transform into equalities. Hence, from here we get following

$$\left(R_{11} + \frac{1-\bar{\lambda}}{\bar{\lambda}} D_{11}\right) x_1 + \left(R_{12} + \frac{1-\bar{\lambda}}{\bar{\lambda}} D_{12}\right) x_2 = Lc_1 + D_{11}\eta_1 + D_{12}\eta_2,$$

$$\left(R_{21} - H_1 + \frac{1-\bar{\lambda}}{\bar{\lambda}} D_{21}\right) x_1 + \left(R_{22} - H_2 + \frac{1-\bar{\lambda}}{\bar{\lambda}} D_{22}\right) x_2 = Lc_2 + D_{21}\eta_1 + D_{22}\eta_2 - y_2.$$

Inasmuch as

$$R_{11}x_1 + R_{12}x_2 = lxc_1 \le Lc_1, \quad R_{21}x_1 + R_{22}x_2 = lxc_2 \le Lc_2,$$

$$\frac{1-\bar{\lambda}}{\bar{\lambda}}(D_{11}x_1 + D_{12}x_2) \le D_{11}\eta_1 + D_{12}\eta_2, \quad \frac{1-\bar{\lambda}}{\bar{\lambda}}(D_{21}x_1 + D_{22}x_2) \le D_{21}\eta_1 + D_{22}\eta_2,$$

$$H_1x_1 + H_2x_2 \ge y_2,$$

So here inequalities are true only in that case, when

$$lx = L, \quad \frac{1-\bar{\lambda}}{\bar{\lambda}} Dx = D\eta, \quad H_1x_1 + H_2x_2 = y_2.$$

Inasmuch as

$$\eta - \frac{1-\bar{\lambda}}{\bar{\lambda}} x \ge 0, \quad D\left(\eta - \frac{1-\bar{\lambda}}{\bar{\lambda}} x\right) = 0,$$

So from these terms for the D matrix we get

$$\eta = \frac{1-\bar{\lambda}}{\bar{\lambda}} x.$$

In such a case we have $\xi = \bar{\lambda}^{-1} x.$ This led to unity of the Neyman's ray for a model (7), that answers the pace of growth $\bar{\lambda}^{-1} > 1.$ The assertion is proved.

In future, it is planned to conduct concrete numerical calculations using a real information of Ukraine.

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